

# Functions

Part One

# Outline for Today

- ***What is a Function?***
  - It's more nuanced than you might expect.
- ***Domains and Codomains***
  - Where functions start, and where functions end.
- ***Defining a Function***
  - Expressing transformations compactly.
- ***Special Classes of Functions***
  - Useful types of functions you'll encounter IRL.
- ***Proofs on First-Order Definitions***
  - A key skill.

What is a function?

# ***Motivating Example 1:*** Database Sharding

Take CS244B!  
Distributed Systems

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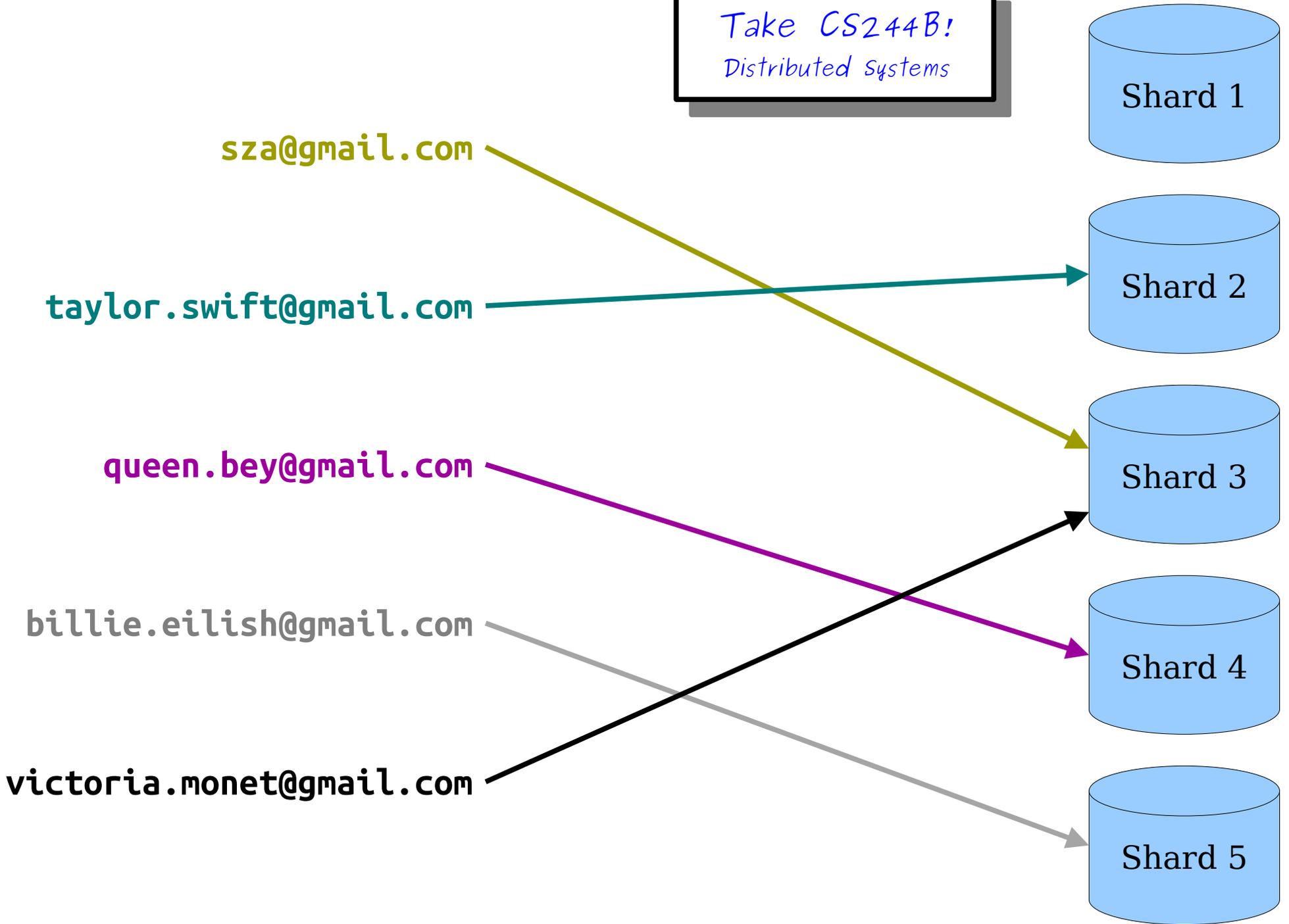
Shard 1

Shard 2

Shard 3

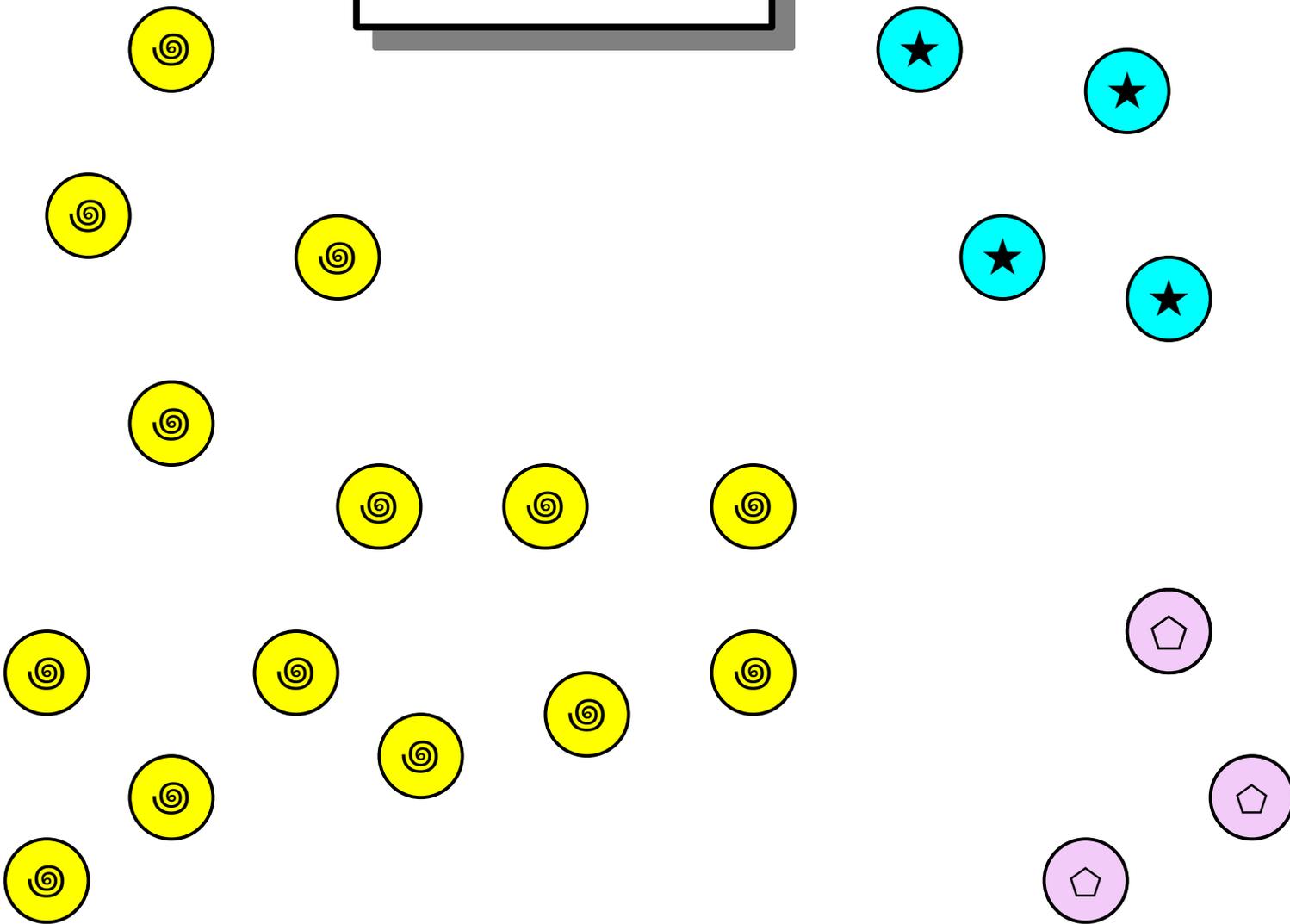
Shard 4

Shard 5



## ***Motivating Example 2:*** Data Clustering

Take CS246!  
Mining Massive Data Sets

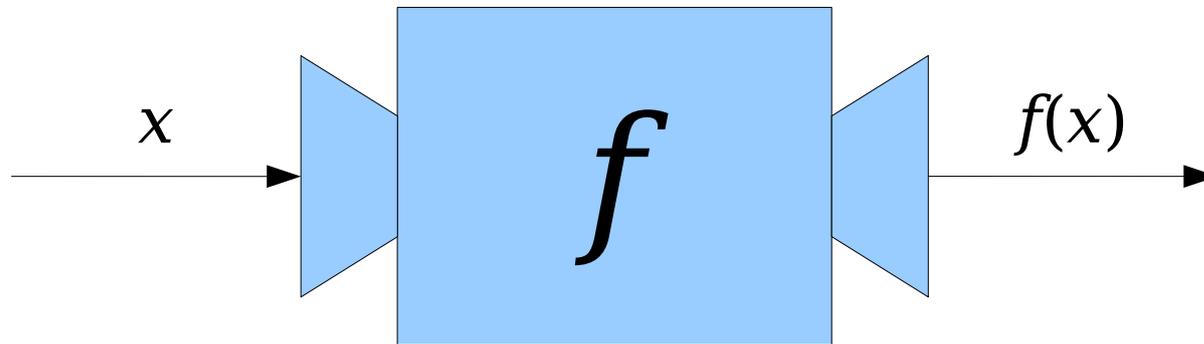


# What's In Common?

- We have a fixed, known set of possible inputs.
  - In our examples: user names and 2D data points
- We have a fixed, known set of possible outputs.
  - In our examples: database shards and cluster labels.
- Each input is assigned an output.
  - Some outputs might be assigned multiple inputs.
  - Some outputs might be assigned no inputs.

## ***High-Level Intuition:***

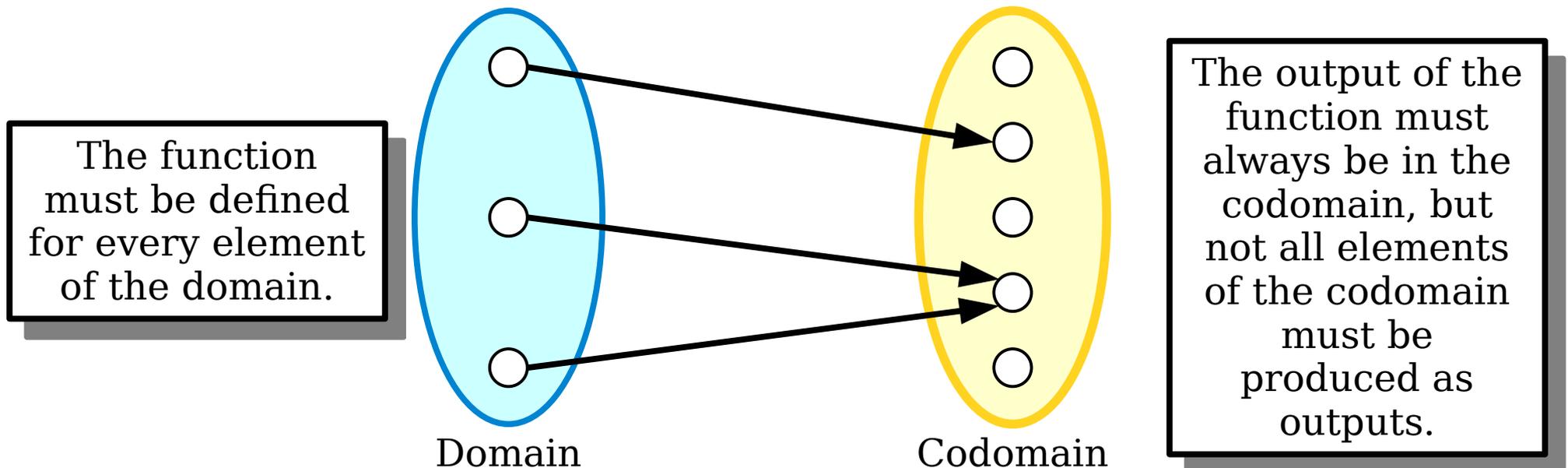
A function is an object  $f$  that takes in exactly one input  $x$  and produces exactly one output  $f(x)$ .



(This is not definition. It's just to help you build and intuition.)

# Domains and Codomains

- Every function  $f$  has two sets associated with it: its **domain** and its **codomain**.
- A function  $f$  can only be applied to elements of its domain. For any  $x$  in the domain,  $f(x)$  belongs to the codomain.



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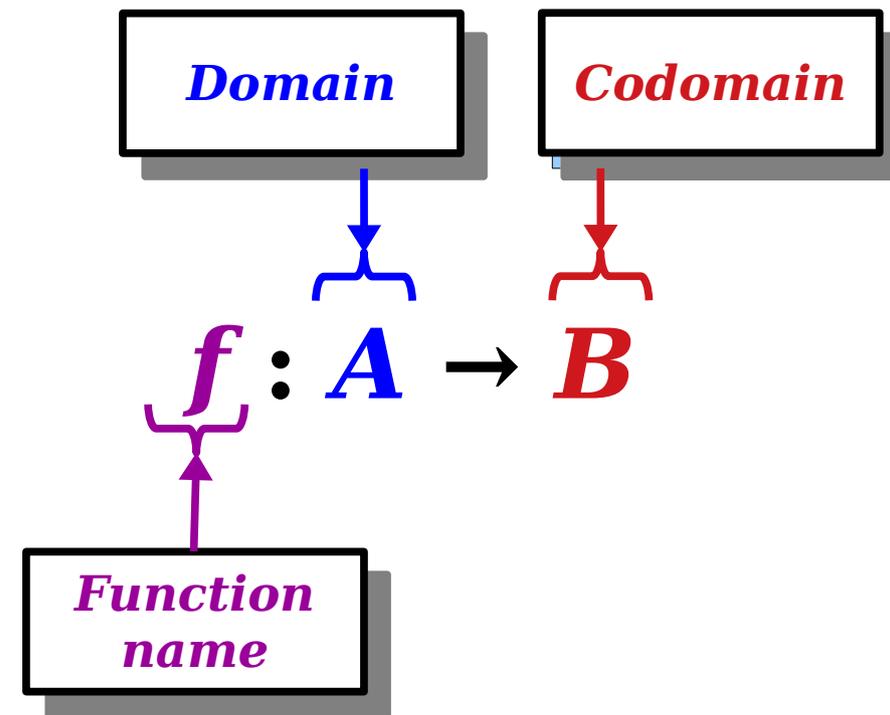
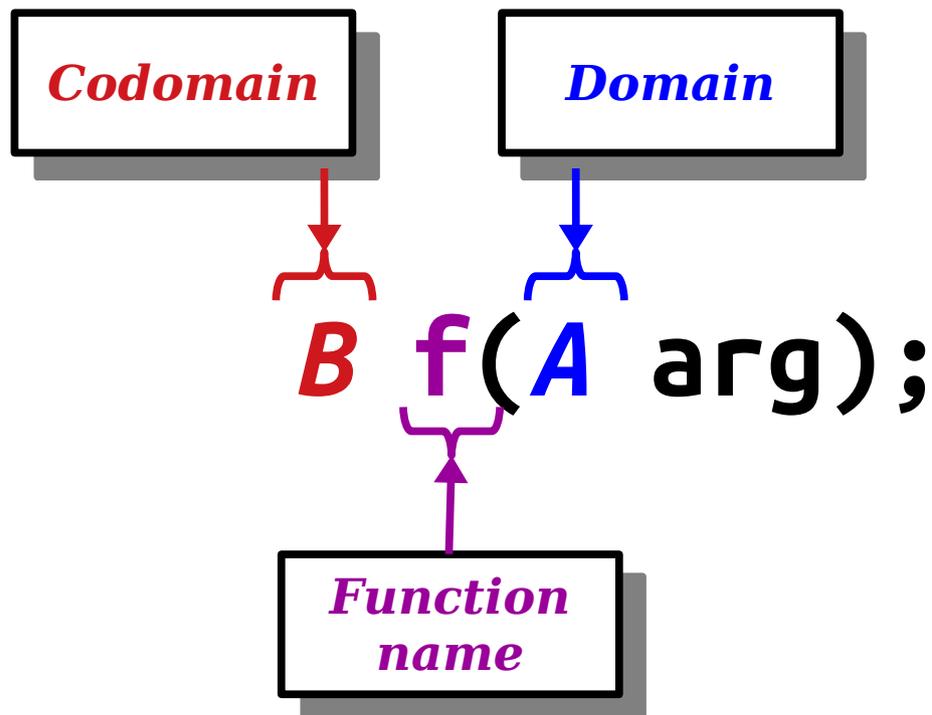
The **domain** of this function is  $\mathbb{R}$ . Any real number can be provided as input.

The **codomain** of this function is  $\mathbb{R}$ . Everything produced is a real number, but not all real numbers can be produced.

```
double absoluteValueOf(double x) {  
    if (x >= 0) {  
        return x;  
    } else {  
        return -x;  
    }  
}
```

# Domains and Codomains

- If  $f$  is a function whose domain is  $A$  and whose codomain is  $B$ , we write  $f : A \rightarrow B$ .
- Think of this like a “function prototype” in C++.



# Some Observations

- Usually, when working with functions, you pick the domain and codomain before defining the rule for the function.
  - Think programming: you usually know what types of things you're working with before you know how they work.
- In mathematics, all functions take in exactly one argument: an element of the domain.
  - If you're clever, you can get two or more arguments to a function while still obeying this rule. Chat with me after class to learn more!
- In mathematics, functions are ***deterministic*** and can't behave randomly.
  - If you're clever, you can get functions that kinda sorta ish look random. Chat with me after class to learn more!

# The Official Rules for Functions

- Formally speaking, we say that  $f : A \rightarrow B$  if the following two rules hold.
- First,  $f$  must obey its domain/codomain rules:

$$\forall a \in A. \exists b \in B. f(a) = b$$

*(“Every input in  $A$  maps to some output in  $B$ .”)*

- Second,  $f$  must be deterministic:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

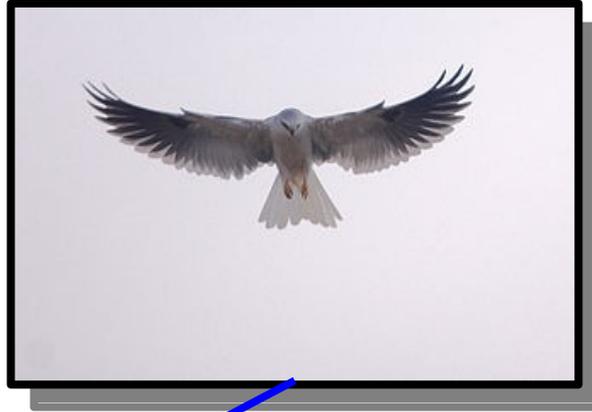
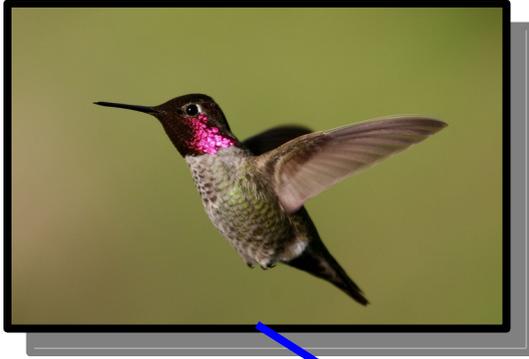
*(“Equal inputs produce equal outputs.”)*

- If you’re ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function have an empty codomain?

# Defining Functions

# Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a **rule** used to evaluate the function.
- All three pieces are necessary.
  - We need to domain to know what the function can be applied to.
  - We need to codomain to know what the output space is.
  - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let's go over a few examples.



*White-Tailed  
Kite*

*Anna's  
Hummingbird*

*Red-Shouldered  
Hawk*

Functions can be defined as a ***picture***.  
Draw the domain and codomain explicitly.  
Then, add arrows to show the outputs.

$$f : \mathbb{Z} \rightarrow \mathbb{Z}, \text{ where}$$
$$f(x) = x^2 + 3x - 15$$

---

Functions can be defined as a **rule**.  
Be sure to explicitly state what the  
domain and codomain are!

$f : \mathbb{Z} \rightarrow \mathbb{N}$ , where

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

---

Some rules are given ***piecewise***. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

# Some Nuances

$f : \mathbb{R} \rightarrow \mathbb{R}$ , where

$$f(x) = \frac{x+2}{x+1}$$

Answer at

<https://cs103.stanford.edu/pollev>

This expression isn't defined when  $x = -1$ , so  $f$  isn't defined over its full domain. We therefore don't consider it to be a function.

Is this a function from  $\mathbb{R}$  to  $\mathbb{R}$ ?

$f : \mathbb{N} \rightarrow \mathbb{R}$ , where

$$f(x) = \frac{x+2}{x+1}$$

Answer at

<https://cs103.stanford.edu/pollev>

Yep, it's a function! Every natural number maps to some real number.

Is this a function from  $\mathbb{N}$  to  $\mathbb{R}$ ?

**Time-Out for Announcements!**

# Problem Set One Solutions

- We've just posted solutions to Problem Set One. They're linked from the main PS1 page.
- We recommend you read over our solution set before finishing PS2.
  - You'll get to see examples of polished written proofs.
  - Each problem has a "Why We Asked This Question" section, which gives some context.
  - We may have solved the problem differently than you, and this will give you more perspectives to use.
- We'll aim to have PS1 graded and returned Wednesday morning / afternoon.
- Please tag pages when submitting PDFs to Gradescope.
- Regret Clause deadline is Tuesday.

Back to CS103!

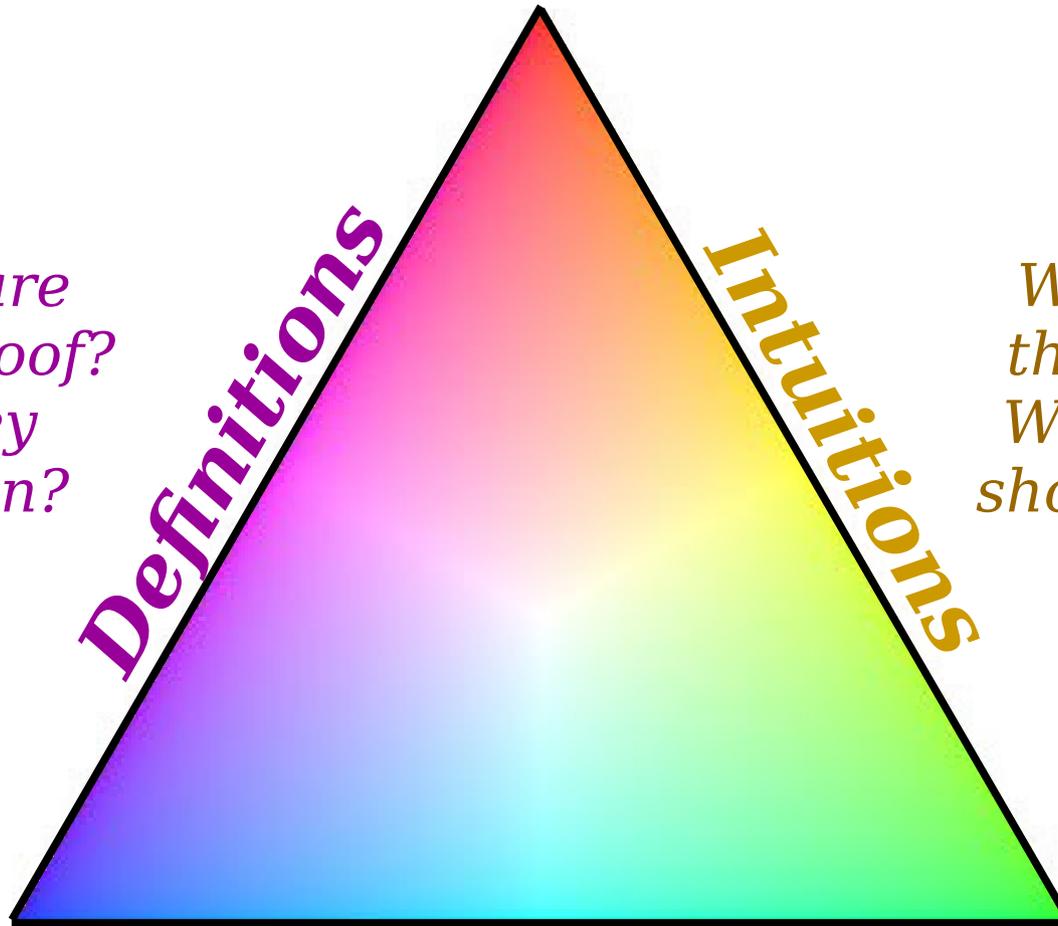
# Special Types of Functions

*What terms are  
used in this proof?  
What do they  
formally mean?*

***Definitions***

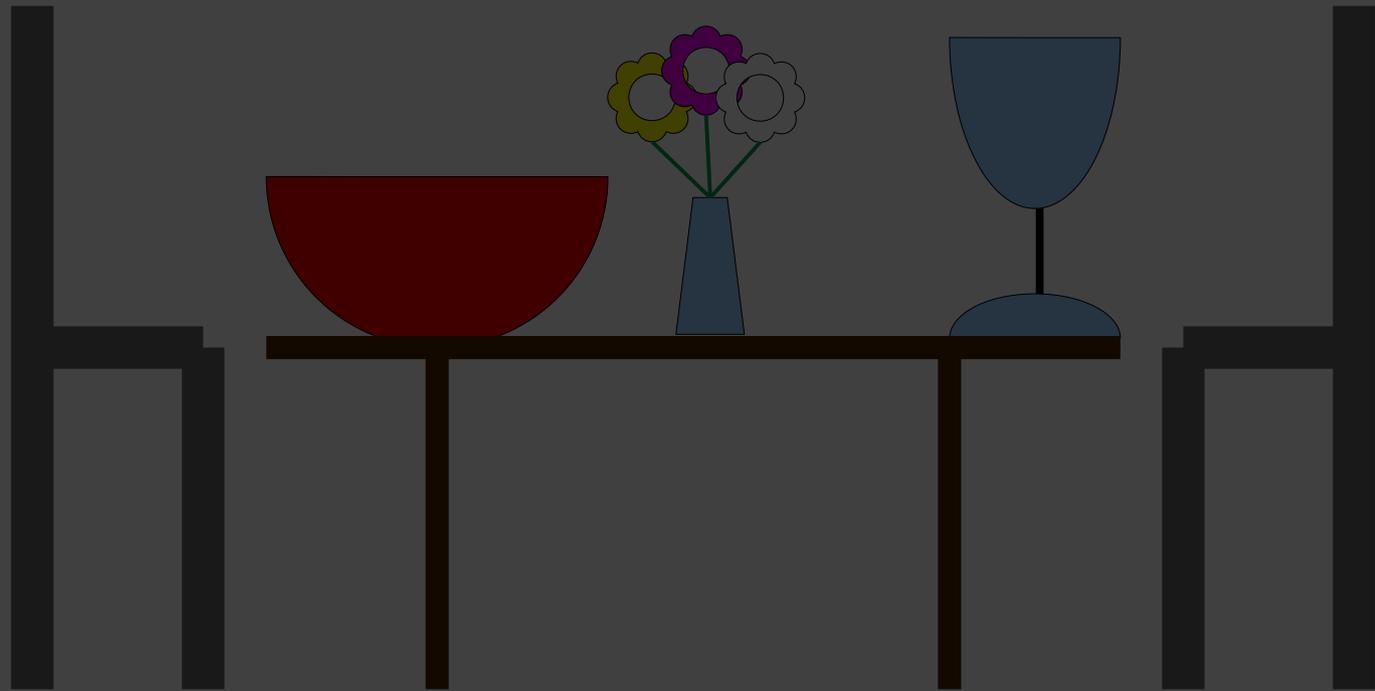
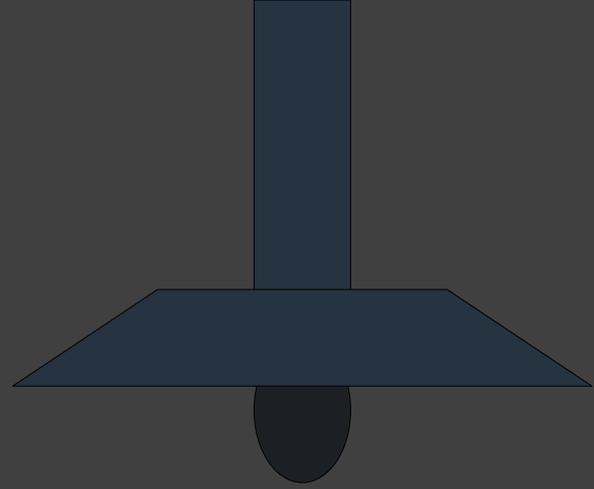
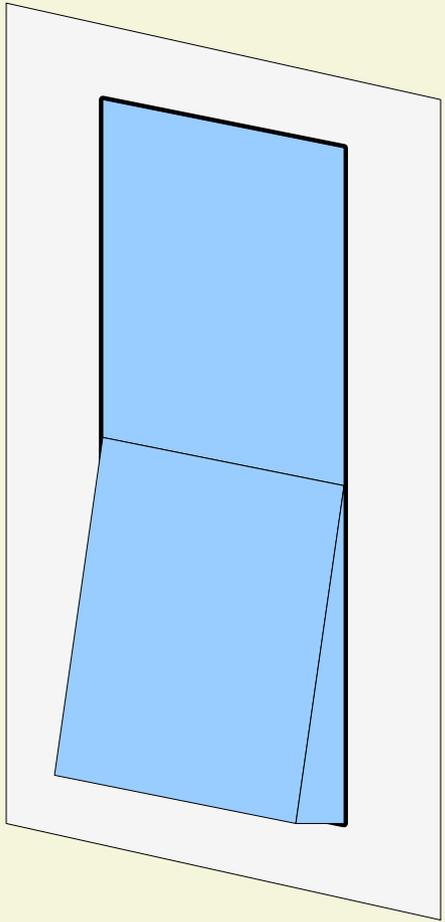
***Intuitions***

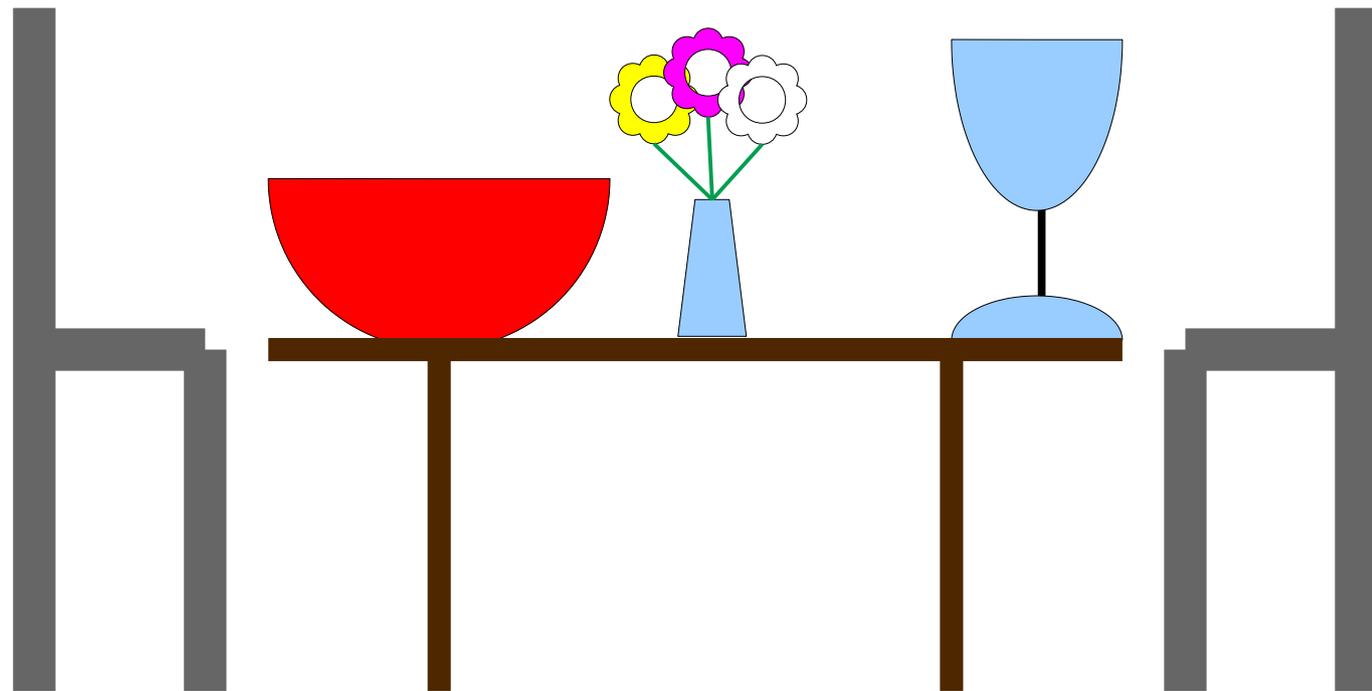
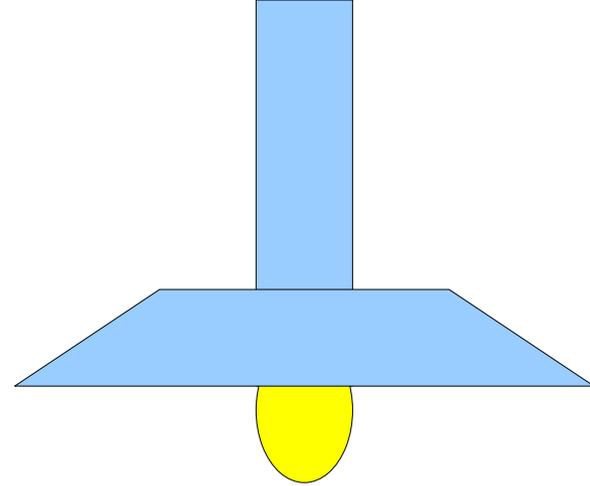
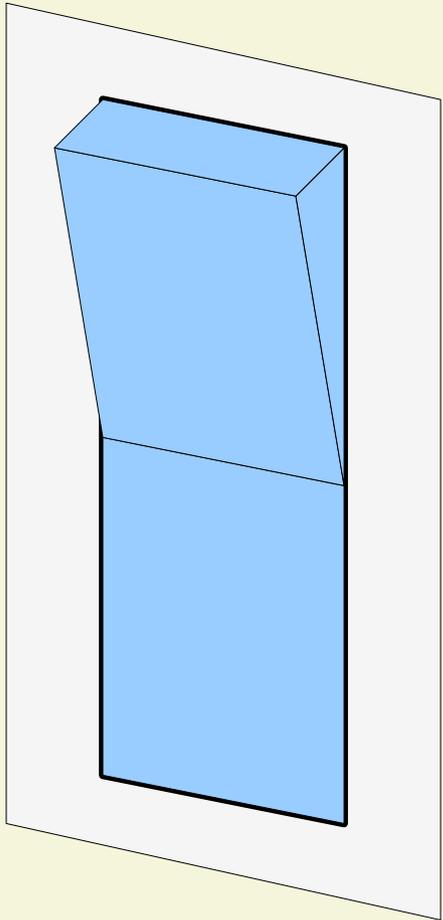
*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*



***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*





# Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic,  $\neg\neg A$  is equivalent to  $A$ .
  - In algebra,  $-(-x) = x$ .
  - In set theory,  $(A \Delta B) \Delta B = A$ . (*Yes, really!*)
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Building encryption systems (symmetric block ciphers).
  - Transmitting a large file to multiple receivers (fountain codes).

# Involutions

- A function  $f : A \rightarrow A$  from a set back to itself is called an ***involution*** when the following first-order logic statement is true about  $f$ :

$$\forall x \in A. f(f(x)) = x.$$

*(“Applying  $f$  twice is equivalent to not applying  $f$  at all.”)*

- Involutions have lots of interesting properties. Let’s explore them and see what we can find.

This is the formal definition. Use it in proofs.

This is just an intuition. Don’t use it in proofs.

# Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined as  $f(x) = x$ . *Yep!*
  - $g : \mathbb{Z} \rightarrow \mathbb{Z}$  defined as  $g(x) = -x$ . *Yep!*
  - $h : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $h(x) = 1/x$ . *Not a function!*
  - $p : \mathbb{N} \rightarrow \mathbb{N}$  defined as follows: *Yep!*

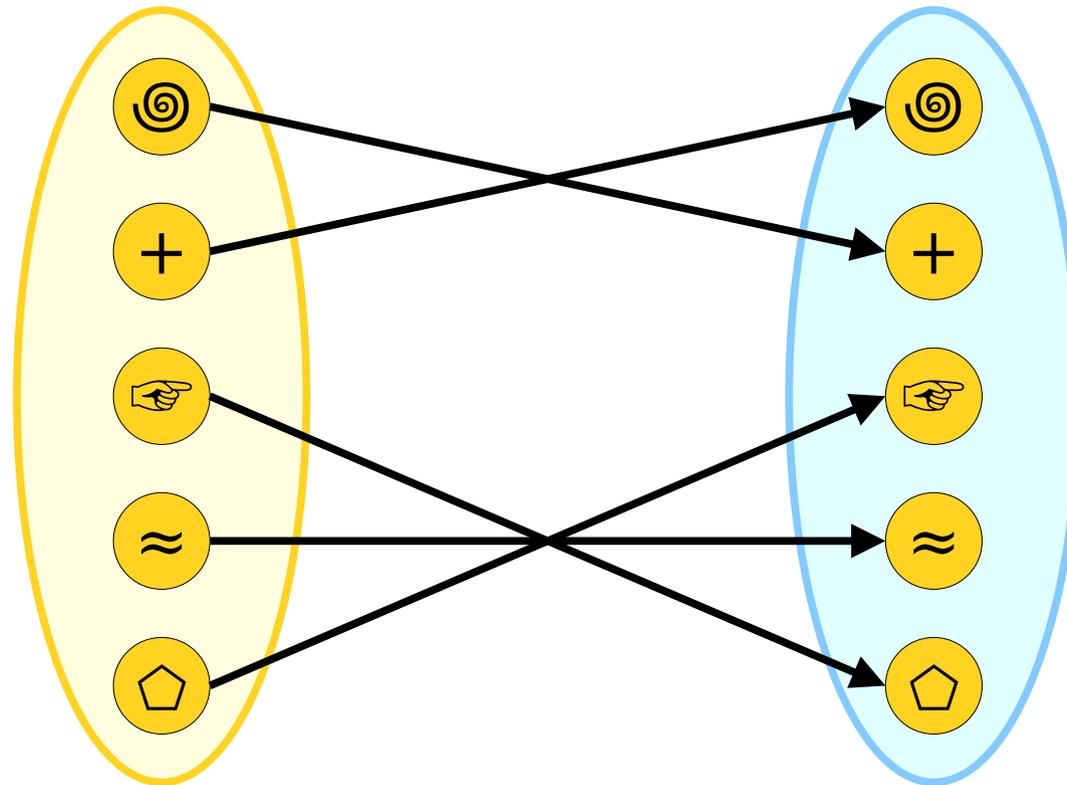
$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

---

A function  $f : A \rightarrow A$  is called an ***involution*** if the following first-order logic statement is true about  $f$ :

$$\forall x \in A. f(f(x)) = x.$$

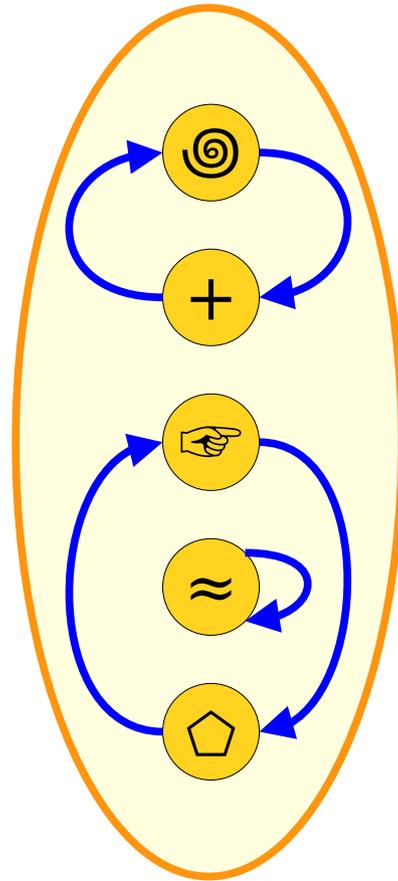
# Involutions, Visually



A function  $f : A \rightarrow A$  is called an ***involution*** if the following first-order logic statement is true about  $f$ :

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# Involutions, Visually



A function  $f : A \rightarrow A$  is called an ***involution*** if the following first-order logic statement is true about  $f$ :

$$\forall x \in A. f(f(x)) = x.$$

# Proofs on Involutions

**Theorem:** The function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

**Proof:**

What does it mean for  $f$  to be an involution?

$$\forall n \in \mathbb{Z}. f(f(n)) = n.$$

Therefore, we'll have the reader pick some  $n \in \mathbb{Z}$ , then argue that  $f(f(n)) = n$ .

**Theorem:** The function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

**Proof:** Pick some  $n \in \mathbb{Z}$ . We need to show that  $f(f(n)) = n$ . To do so, we consider two cases.

*Case 1:*  $n$  is even. Then  $f(n) = n+1$ , which is odd. This means that  $f(f(n)) = f(n+1) = (n+1) - 1 = n$ .

*Case 2:*  $n$  is odd. Then  $f(n) = n - 1$ , which is even. Then we see that  $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$ .

In either case, we see that  $f(f(n)) = n$ , which is what we need to show. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

**Theorem:** The function  $f : \mathbb{N} \rightarrow \mathbb{N}$  defined as  $f(n) = n^2$  is not an involution.

What does it mean for  $f$  to be an involution?

$$\forall n \in \mathbb{N}. f(f(n)) = n.$$

What is the negation of this statement?

$$\begin{aligned} &\neg \forall n \in \mathbb{N}. f(f(n)) = n \\ &\exists n \in \mathbb{N}. \neg (f(f(n)) = n) \\ &\exists n \in \mathbb{N}. f(f(n)) \neq n \end{aligned}$$

Therefore, we need to find some concrete choice of  $n$  such that  $f(f(n)) \neq n$ .

**Theorem:** The function  $f : \mathbb{N} \rightarrow \mathbb{N}$  defined as  $f(n) = n^2$  is not an involution.

**Proof:** We need to show that there is some  $n \in \mathbb{N}$  where  $f(f(n)) \neq n$ .

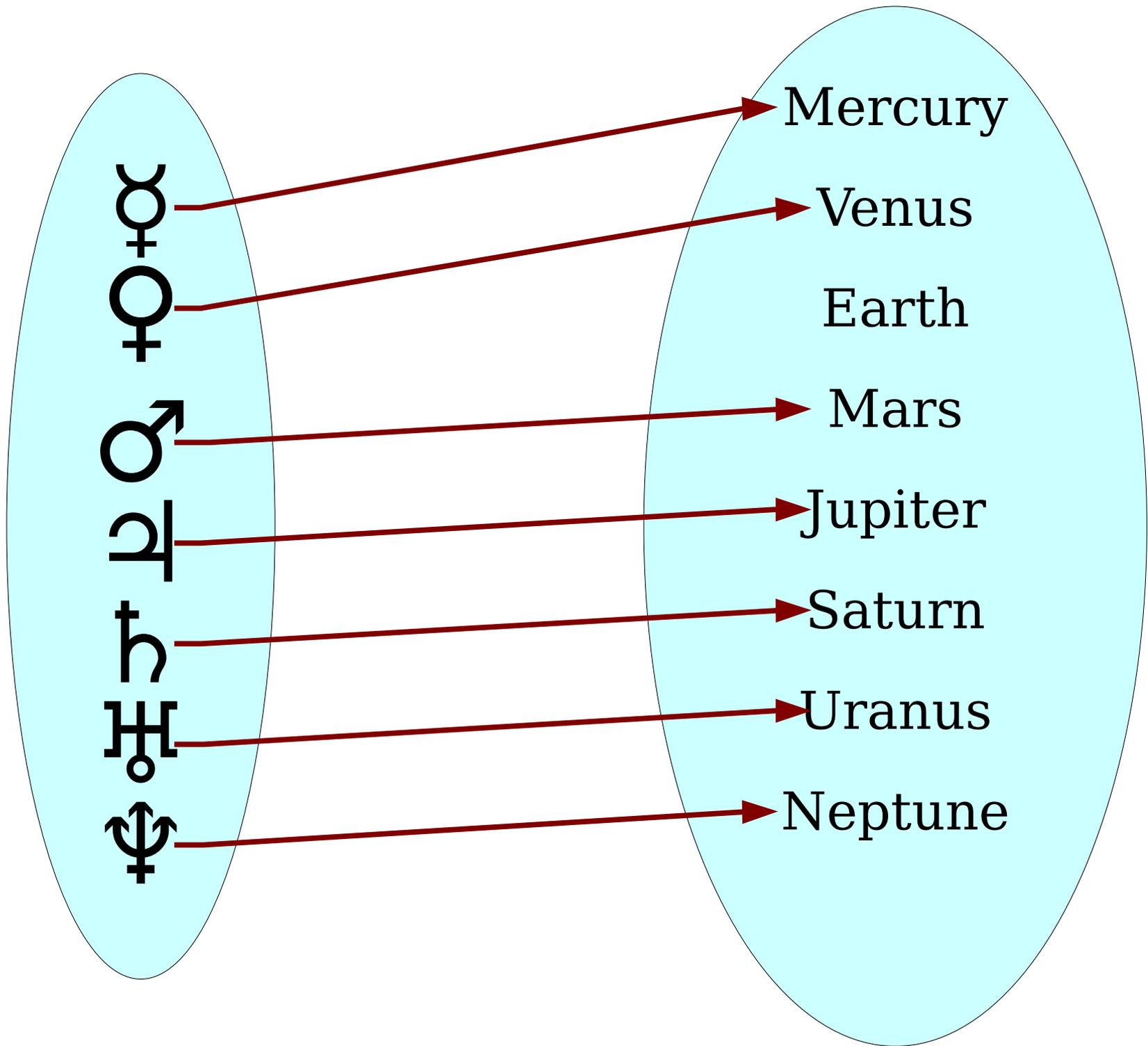
Pick  $n = 2$ . Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

which means that  $f(f(n)) \neq n$ , as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

# Another Class of Functions



# Injective Functions

- A function  $f : A \rightarrow B$  is called **injective** (or **one-to-one**) when the following statement is true about  $f$ :

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

*(“If the inputs are different, the outputs are different.”)*

- The following first-order definition is equivalent (*why?*) and is often useful in proofs.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

*(“If the outputs are the same, the inputs are the same.”)*

- A function with this property is called an **injection**.
- How does this compare to our second rule for functions?

# Injective

- Let  $S$  be the set of all CS103 students. Which of the following are injective?
  - $f: S \rightarrow \mathbb{N}$  where  $f(x)$  is  $x$ 's Stanford ID number.
  - $g: S \rightarrow C$ , where  $C$  is the set of all continents and  $g(x)$  is  $x$ 's continent of birth.
  - $h: S \rightarrow N$ , where  $N$  is the set of all given (first) names, where  $h(x)$  is  $x$ 's given (first) name.

---

$f: A \rightarrow B$  is **injective** when either equivalent statement is true:

$$\forall x_1 \in A. \forall x_2 \in A. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\forall x_1 \in A. \forall x_2 \in A. (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$$

# Proofs on Injections

# Injective Functions

**Theorem:** Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be defined as  $f(n) = 2n + 7$ .  
Then  $f$  is injective.

**Proof:**

What does it mean for the function  $f$  to be injective?

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. ( f(n_1) = f(n_2) \rightarrow n_1 = n_2 )$$

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. ( n_1 \neq n_2 \rightarrow f(n_1) \neq f(n_2) )$$

Therefore, we'll pick arbitrary  $n_1, n_2 \in \mathbb{N}$ ,  
assume  $f(n_1) = f(n_2)$ , then prove that  $n_1 = n_2$ .

# Injective Functions

**Theorem:** Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be defined as  $f(n) = 2n + 7$ .  
Then  $f$  is injective.

**Proof:** Consider any  $n_1, n_2 \in \mathbb{N}$  where  $f(n_1) = f(n_2)$ . We will prove that  $n_1 = n_2$ .

Since  $f(n_1) = f(n_2)$ , we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so  $n_1 = n_2$ , as required. ■

Good exercise: Repeat this proof using the other definition of injectivity!

# Injective Functions

**Theorem:** Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be defined as  $f(n) = 2n + 7$ .  
Then  $f$  is injective.

**Proof:** Consider any  $n_1, n_2 \in \mathbb{N}$  where  $f(n_1) = f(n_2)$ . We will prove that  $n_1 = n_2$ .

Since  $f(n_1) = f(n_2)$ , we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so  $n_1 = n_2$ , as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

# Injective Functions

**Theorem:** Let  $f : \mathbb{Z} \rightarrow \mathbb{N}$  be defined as  $f(x) = x^4$ . Then  $f$  is not injective.

**Proof:**

What does it mean for  $f$  to be injective?

$$\forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg \forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \neg \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. \neg (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. (x_1 \neq x_2 \wedge \neg (f(x_1) \neq f(x_2)))$$

$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. (x_1 \neq x_2 \wedge f(x_1) = f(x_2))$$

Therefore, we need to find  $x_1, x_2 \in \mathbb{Z}$  such that  $x_1 \neq x_2$ , but  $f(x_1) = f(x_2)$ .

Can we do that?

# Injective Functions

**Theorem:** Let  $f : \mathbb{Z} \rightarrow \mathbb{N}$  be defined as  $f(x) = x^4$ . Then  $f$  is not injective.

**Proof:** We will prove that there exist integers  $x_1$  and  $x_2$  such that  $x_1 \neq x_2$ , but  $f(x_1) = f(x_2)$ .

Let  $x_1 = -1$  and  $x_2 = +1$ . Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so  $f(x_1) = f(x_2)$  even though  $x_1 \neq x_2$ , as required. ■

		To <b>prove</b> that this is true...
$\forall x. A$		Have the reader pick an arbitrary $x$ . We then prove $A$ is true for that choice of $x$ .
$\exists x. A$		Find an $x$ where $A$ is true. Then prove that $A$ is true for that specific choice of $x$ .
$A \rightarrow B$		Assume $A$ is true, then prove $B$ is true.
$A \wedge B$		Prove $A$ . Also prove $B$ .
$A \vee B$		Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$ . <i>(Why does this work?)</i>
$A \leftrightarrow B$		Prove $A \rightarrow B$ and $B \rightarrow A$ .
$\neg A$		Simplify the negation, then consult this table on the result.

# Two More Classes of Functions

Lassen Peak

Mt. Shasta

Crater Lake

Mt. McLoughlin

Mt. Hood

Mt. St. Helens

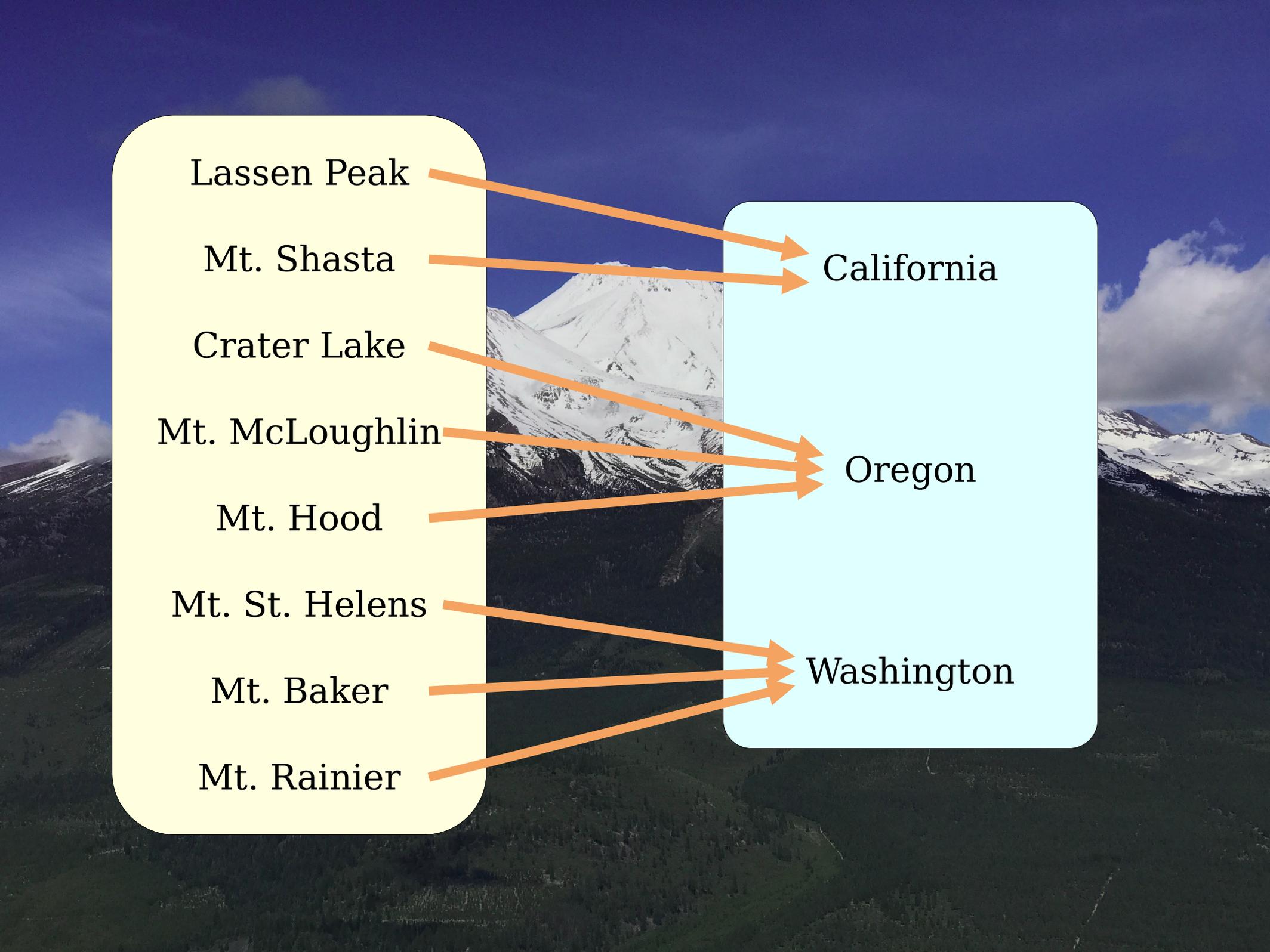
Mt. Baker

Mt. Rainier

California

Oregon

Washington



# Surjective Functions

- A function  $f : A \rightarrow B$  is called **surjective** (or **onto**) when this first-order logic statement is true about  $f$ :

$$\forall b \in B. \exists a \in A. f(a) = b$$

*(“For every possible output, there's an input that produces it.”)*

- A function with this property is called a **surjection**.
- How does this compare to our first rule of functions?

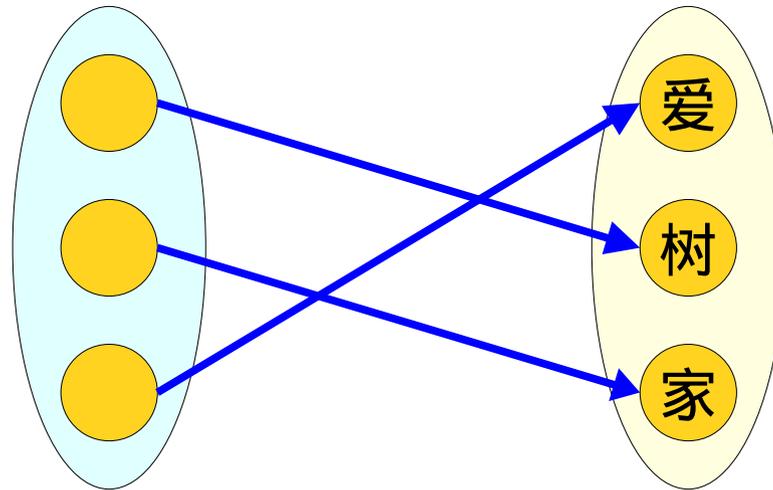
Check the appendix for  
sample proofs involving  
surjections.

# Injections and Surjections

- An injective function associates *at most* one element of the domain with each element of the codomain.
- A surjective function associates *at least* one element of the domain with each element of the codomain.
- What about functions that associate *exactly one* element of the domain with each element of the codomain?

# Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if  $f : A \rightarrow B$  is a bijection, then  $f$  represents a way of pairing off elements of  $A$  and elements of  $B$ .



# Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = x$ . *Yep!*
  - $f : \mathbb{Z} \rightarrow \mathbb{R}$  defined as  $f(x) = x$ . *Nope!*
  - $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = 2x + 1$ . *Yep!*
  - $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined as  $f(x) = 2x + 1$ . *Nope!*

---

A ***bijection*** is a function that is both injective and surjective.

# Next Time

- ***First-Order Assumptions***
  - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
  - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.

## ***Appendix:*** More Proofs on Functions

***Proof 1:*** Proving a function is surjective.

# Surjective Functions

**Theorem:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = 2x$ . Then  $f(x)$  is surjective.

**Proof:**

What does it mean for  $f$  to be surjective?

$$\forall y \in \mathbb{R}. \exists x \in \mathbb{R}. f(x) = y$$

Therefore, we'll choose an arbitrary  $y \in \mathbb{R}$ , then prove that there is some  $x \in \mathbb{R}$  where  $f(x) = y$ .

# Surjective Functions

**Theorem:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = 2x$ . Then  $f(x)$  is surjective.

**Proof:** Consider any  $y \in \mathbb{R}$ . We will prove that there is a choice of  $x \in \mathbb{R}$  such that  $f(x) = y$ .

Let  $x = y / 2$ . Then we see that

$$f(x) = f(y / 2) = 2y / 2 = y.$$

So we see that  $f(x) = y$ , as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

***Proof 2:*** Proving a function is not surjective.

# Surjective Functions

**Theorem:** Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be defined as  $g(n) = 2n$ . Then  $g(x)$  is not surjective.

What does it mean for  $g$  to be surjective?

$$\forall n \in \mathbb{N}. \exists m \in \mathbb{N}. g(m) = n$$

What is the negation of the above statement?

$$\neg \forall n \in \mathbb{N}. \exists m \in \mathbb{N}. g(m) = n$$

$$\exists n \in \mathbb{N}. \neg \exists m \in \mathbb{N}. g(m) = n$$

$$\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. g(m) \neq n$$

Therefore, we need to find a natural number  $n$  where, regardless of which  $m$  we pick, we have  $g(m) \neq n$ .

# Surjective Functions

**Theorem:** Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be defined as  $g(n) = 2n$ . Then  $g(x)$  is not surjective.

**Proof:** Let  $n = 137$ . Now, pick an arbitrary  $m \in \mathbb{N}$ . We need to show that  $g(m) \neq n$ .

Notice that  $g(m) = 2m$  is even, while 137 is odd. Therefore, we have  $g(m) \neq 137$ , as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.